Ex 1 Let $E = \mathbb{R}^2$, show that E is not a vector over $\mathbb{R}$ under the following addition and scalar multiplication:

(i) $(a,b) + (c,d) = (a+d, b)$ and $\alpha(a,b) = (\alpha a, \alpha b)$

(ii) $(a,b) + (c,d) = (a+c, b+d)$ and $\alpha(a,b) = (\alpha a + 2\alpha b, \alpha a + \alpha b)$

(iii) $(a,b) + (c,d) = (a+c, b+d)$ and $\alpha(a,b) = (\alpha^2 a, \alpha^2 b)$.

solution (i) Assume that E is a vector space over $\mathbb{R}$

then $(1,1) + (1,1) = 2(1,1)$

But $(1,1) + (1,1) = (1+1, 1)$ and $2(1,1) = (2,2)$

hence $(2,1) = (2,2)$, which is impossible

therefore E is not a vector space over $\mathbb{R}$.

(ii) Assume that E is a vector space over $\mathbb{R}$.

then $(1,1) + (1,1) = 2(1,1)$

But $(1,1) + (1,1) = (1+1, 1+1)$ and $2(1,1) = (2+2, 2+4)$

hence $(2,2) = (4,6)$, which is impossible

therefore E is not a vector space over $\mathbb{R}$.

(iii) Assume that E is a vector space over $\mathbb{R}$, then

$$(1,1)+(1,1) = 2(1,1)$$

But $(1,1)+(1,1) = (1+1, 1+1)$ and $2(1,1) = (2,4)$ hence
$(2,2) = (2,4)$ which is impossible

Therefore E is not a vector space over $\mathbb{R}$.

<u>Ex2</u> Say if W is or is not a subspace of $\mathbb{R}^3$ over $\mathbb{R}$ in the following cases:

(i) $W = \{(a,b,c) \in \mathbb{R}^3 ;\ c \neq 0\}$

(ii) $W = \{(a,b,c) \in \mathbb{R}^3,\ a+b+5c = 0\}$

(iii) $W = \{(a,b,c) \in \mathbb{R}^3,\ 2a^2 - b^2 + c^2 = 0\}$

(iv) $W = \{(a,b,c) \in \mathbb{R}^3 ;\ b \geq 0\}$.

<u>solution</u>: (i) Assume that W is a subspace of $\mathbb{R}^3$ over $\mathbb{R}$, then $(0,0,0) \in W$, which is impossible hence W is not a subspace of $\mathbb{R}^3$ over $\mathbb{R}$.

(ii) We have $0+0+5\times 0 = 0$, hence $(0,0,0) \in W$ and so $W \neq \phi$.

Let $x = (a, b, c) \in W$ and $x' = (a', b', c') \in W$ and $\alpha \in K$, then $x + x' = (a+a', b+b', c+c')$

and $\alpha x = (\alpha a, \alpha b, \alpha c)$

But $a + b + 5c = 0$ and $a' + b' + 5c' = 0$

hence $(a+a') + (b+b') + 5(c+c') = a + b + 5c + a' + b' + 5c'$

$= 0 + 0 = 0$

and $\alpha a + \alpha b + 5(\alpha c) = \alpha(a + b + 5c) = \alpha 0 = 0.$

and so $x + x' \in W$ and $\alpha x \in W$

then W is a subspace of $\mathbb{R}^3$ over $\mathbb{R}$

(iii) Assume that W is a subspace of $\mathbb{R}^3$ over $\mathbb{R}$.

then as $(0,1,1) \in W$ and $(1,1,0) \in W$ we obtain

that $(0,1,1) + (1,1,0) \in W$ hence $(1,2,1) \in W$ and

so $(1)^2 - (2)^2 + (1)^2 = 0$ which give $-2 = 0$, impossible

Thus W is not a subspace of $\mathbb{R}^3$ over $\mathbb{R}$

(iv) Assume that W is a subspace of $\mathbb{R}^3$ of $\mathbb{R}$

then as $(0,1,1) \in W$ and $-2 \in \mathbb{R}$, we obtain

that $-2(0,1,1) \in W$ hence $(0,-2,-2) \in W$ which give

that $-2 \geq 0$ impossible, Thus W is not subspace of $\mathbb{R}^3$ over $\mathbb{R}$

Ex 3 Let a be a real positive number and $I = [-a, +a]$

Let E be the set of all mappings of I to $\mathbb{R}$

Say if w is or is not a subspace of E over $\mathbb{R}$ in the following cases:

(i) $w = \{f \in E,\ f(3) = 0\}$. with $a \geqslant 3$

(ii) $w = \{f \in E,\ f(2) = 0\}$ with $a \geqslant 2$

(iii) $w = \{f \in E,\ f(1) = f(3)\}$. with $a \geqslant 3$

Solution (i) We have $0_E(3) = 0$ hence $0_E \in w$ and so $w \neq \emptyset$.

Let $f, g \in w$ and $\alpha \in \mathbb{R}$, then $f(3) = 0$ and $g(3) = 0$

hence $(f+g)(3) = f(3) + g(3) = 0 + 0 = 0$

and $(\alpha f)(3) = \alpha(f(3)) = \alpha \cdot 0 = 0$

and so $f+g \in w$ and $\alpha f \in w$

whence w is a subspace of E over $\mathbb{R}$.

(ii) Assume that w is a subspace of E over $\mathbb{R}$

consider the mapping $f : I \longrightarrow \mathbb{R}$ defined by $f(x) = 1$, then $f \in w$ hence $2f \in w$, and so

$(2f)(2) = 1$, whence $2(f(2)) = 1$ which gives $2 = 1$ impossible. thus W is not a subspace of E over $\mathbb{R}$.

(ii) We have $0_E(1) = 0 = 0_E(3)$ hence $0_E \in W$ and so $W \neq \phi$

Let $f, g \in W$ and $\alpha \in \mathbb{R}$ then $f(1) = f(3)$ and $g(1) = g(3)$ hence

$$(f+g)(1) = f(1) + g(1) = f(3) + g(3) = (f+g)(3)$$

and $(\alpha f)(1) = \alpha(f(1)) = \alpha(f(3)) = (\alpha f)(3)$

and so $f+g \in W$ and $\alpha f \in W$ then W is a subspace of E over $\mathbb{R}$.

Ex4 Show that W is a subspace of $M_n(K)$ over K in the following cases:

(i) W = the set of matrices of $M_n(K)$ of trace zero

(ii) W = the set of symetric matrices of $M_n(K)$

(iii) W = the set of upper triangular matrices of $M_n(K)$

Solution (i) We have $tr(0)=0$, hence $0 \in W$ and so $W \neq \phi$, Let $X, Y \in W$ and $a \in K$, then $tr(X)=0$ and $tr(Y)=0$ hence $tr(X+Y)=tr(X)+tr(Y)=0+0=0$

and $tr(aX)=a\,tr(X)=a.0=0$

and so $X+Y \in W$ and $aX \in W$ then W is subspace of $M_n(K)$

(ii) we have $t_0=0$ hence $0 \in W$, and so $W \neq \phi$

Let $X, Y \in W$ and $a \in K$ then $t_X = X$ and $t_Y = Y$

hence $t(X+Y) = t_X + t_Y = X+Y$

or $(X+Y)^t = X^t + Y^t = X+Y$

and $(aX)^t = aX^t = aX$

and so $X+Y \in W$ and $aX \in W$ whence W is subspace of $M_n(K)$.

(iii) set $0=(b_{ij})$ $b_{ij}=0$ $\forall i>j$ then $0 \in W$ and so $W \neq \phi$. Let $X, Y \in W$ and $a \in K$. set

$X=(x_{ij})$, $Y=(y_{ij})$, $X+Y=(c_{ij})$ and $aX=(d_{ij})$

then $x_{ij}=0$ and $y_{ij}=0$ $\forall i>j$

hence $\forall i>j$ ~~we~~ we have $c_{ij}=x_{ij}+y_{ij}=0+0=0$

and $d_{ij}=a x_{ij}=a.0=0$

and so $X+Y \in W$ and $aX \in W$, whence W is a subspace of $M_n(K)$.

Ex5 Find the real a so the element $u=(a,-1,3) \in \mathbb{R}^3$ is a linear combination over $\mathbb{R}$ of the elements $x=(3,0,-2)$ and $y=(2,-1,-5)$

solution We have $u=(a,-1,3)$ is a linear combination over $\mathbb{R}$ of the elements $x=(3,0,-2)$ and $y=(2,-1,5)$ so that there exist $b,c \in \mathbb{R}$ such that

$$u = bx + cy$$

But $u=bx+cy \Leftrightarrow (a,-1,3) = b(3,0,-2)+c(2,-1,-5)$

$\Leftrightarrow (a,-1,3) = (3b,0,-2b) + (2c,-c,-5c)$

$\Leftrightarrow (a,-1,3) = (3b+2c, -c, -2b-5c)$

$\Leftrightarrow 3b+2c=a$; $-c=-1 \Leftrightarrow \boxed{c=1}$ $\quad -2b-5c=3$

$-2b-5=3$

$\Leftrightarrow 3(-4)+2(1)=a \Rightarrow \boxed{a=-10}$ $\quad -2b=8 \Rightarrow \boxed{b=-4}$

hence u is a linear combination of x and y over $\mathbb{R}$ if and only if $a=-10$.

Ex6 show that the element $x = (1,5,-1)$ of $\mathbb{R}^3$ is a linear combination over $\mathbb{R}$ of the elements
$u = (3,2,1)$ $v = (0,1,-2)$ and $w = (0,0,1)$

Solution $x = au + bv + cw \Leftrightarrow (1,5,-1) = a(3,2,1) + b(0,1,-2) + c(0,0,1)$

$\Leftrightarrow (1,5,-1) = (3a, 2a, a) + (0, b, -2b) + (0,0,c)$

$\Leftrightarrow (1,5,-1) = (3a, 2a+b, a-2b+c)$

$\Leftrightarrow 3a = 1 \Rightarrow \boxed{a = \frac{1}{3}}$ $2a+b = 5 \Rightarrow b = 5 - \frac{2}{3} = \boxed{\frac{13}{3}}$

$c = -1 - a + 2b = \boxed{\frac{22}{3}}$

hence $x = \frac{1}{3}u + \frac{13}{3}v + \frac{22}{3}w$.

Ex7 write the element $u = (1,-2,5)$ of $\mathbb{R}^3$ as a linear combination over $\mathbb{R}$ of the elements x_1, x_2, and x_3 in the following cases.

(i) $x_1 = (1,1,1)$ $x_2 = (0,1,1)$ and $x_3 = (0,0,1)$.

(ii) $x_1 = (1,0,2)$ $x_2 = (2,1,3)$ and $x_3 = (0,1,1)$

Solution Let's find $a, b, c \in \mathbb{R}$ such that $u = ax_1 + bx_2 + cx_3$

$\Leftrightarrow (1,-2,5) = a(1,1,1) + b(0,1,1) + c(0,0,1)$

$\Leftrightarrow (1,-2,5) = (a, a+b, a+b+c)$

$\Leftrightarrow a = 1$ $b = -2 - a = -3$ $\boxed{c = 7} \Leftrightarrow u = x_1 - 3x_2 + 7x_3$

(ii) $u = ax_1 + bx_2 + cx_3 \Leftrightarrow (1,-2,5) = a(1,0,2) + b(2,1,3) + c(0,1,1)$

$\Leftrightarrow (1,-2,5) = (a+2b,\ b+c,\ 2a+3b+c)$

$\Leftrightarrow a+2b=1 \qquad b+c=-2 \qquad 2a+3b+c=5$

the augmented matrix of this system is:

$$A = \left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & -2 \\ 2 & 3 & 1 & 5 \end{array}\right] \xrightarrow[\text{REF}]{} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 2 & 1 \end{array}\right]$$

~~we hadh~~

$a+2b=1 \qquad b+c=-2 \qquad 2c=1$

$\Rightarrow c = \frac{1}{2} \qquad b = -\frac{5}{2} \qquad a = 6$

So $\quad u = 6x_1 - \frac{5}{2}x_2 + \frac{1}{2}x_3$..

<u>Ex 8</u> Let $U = \{(a,b,c) \in \mathbb{R}^3;\ a+b=0\}$
and $V = \{(a,b,c) \in \mathbb{R}^3;\ a+b+2c=0 \text{ and } b=c\}$

1) show that U and V are two subspaces of $\mathbb{R}^3$ over $\mathbb{R}$.

2) Find a systems of generators of U, $U+V$ and $U \cap V$.

3) show that the system of generators of $U+V$ is a system of generators of $\mathbb{R}^3$

4) deduce $\mathbb{R}^3 = U \oplus V$.

Ex 8 solution

$U = \{(a,b,c) \in \mathbb{R}^3 \; ; a+b=0\}$
$V = \{(a,b,c) \in \mathbb{R}^3 \; ; a+b+2c=0 \text{ and } b=c\}$

1) U subspace ??

• $0 = (0,0,0)$
$\Rightarrow 0+0=0 \quad \Rightarrow 0 \in S \Rightarrow S \neq \{\emptyset\}$

• Let $\vec{u} = (a,b,c) \in U$ with $a+b=0$
$\vec{v} = (a',b',c') \in U$ with $a'+b'=0$

$\vec{u}+\vec{v} = (a+a', b+b', c+c')$

$(a+a') + (b+b') = 0$
$a+b+a'+b' = 0$
$0 + 0 = 0$
$\Rightarrow \vec{u}+\vec{v} \in U$

• Let K a scalar

$K\vec{u} = (Ka, Kb, Kc)$

$Ka + Kb = 0$

$K(a+b) = 0$
$K0 = 0$
$\Rightarrow K\vec{u} \in U$

So U is a subspace of $\mathbb{R}^3$

$V = (a, b, c) \in \mathbb{R}^3 \quad a+b+2c = 0 \ , b = c$

- $0 = (0, 0, 0)$

$0 + 0 + 20 = 0$ and $0 = 0$

$\Rightarrow \quad 0 \in V \quad \Rightarrow \quad V \neq \{\emptyset\}$

- Let $\vec{u} = (a, b, c)$ with $a+b+2c=0 \ ; b=c$

$\vec{v} = (a', b', c')$ with $a'+b'+2c'=0 \ ; b'=c'$

$u+v = (a+a', b+b', c+c')$

$b+b' = c+c'$ ✓

$(a+a') + (b+b') + 2(c+c') = 0$

$(a+b+2c) + (a'+b'+2c') = 0$

$0 + 0 = 0$ ✓

Let K scalar

$K\vec{u} = (Ka, Kb, Kc)$

$Kb = Kc$ ✓

$Ka + Kb + K2c = 0$

$K(a+b+2c) = 0$

$K0 = 0$ ✓

$\Rightarrow V$ is subspace of $\mathbb{R}^3$

Systeme of generator of U:

$U\{(a,b,c) \in \mathbb{R}^3 \; ; a+b=0$

$\Rightarrow a=-b$

$\Rightarrow U = \boxed{(-b,b,c)}$

$(-b,b,c) = b\underbrace{(-1,1,0)} + c\underbrace{(0,0,1)}$

Sys of gen of V:

$V=\{(a,b,c) \in \mathbb{R}^3 \; ; \; a+b+2c=0$

$b=c$

$\Rightarrow a+c+2c=0$

$a+3c=0$

$a=-3c$

$\Rightarrow V \; \boxed{(-3c,c,c)}$

$(-3c,c,c) = c\underbrace{(-3,1,1)}$

Sys of gen of $U \cap V$

$U \cap V = (a, b, c) \in \mathbb{R}^3$ with

$$\begin{cases} a + b = 0 \\ a + b + 2c = 0 \\ b - c = 0 \end{cases} \quad \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$R_2 - R_1 \to R_2$ $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$

$R_2 \longleftrightarrow R_3$ $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right]$

$\frac{R_3}{2}$ $\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$

$R_2 + R_3 \to R_2$ $\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$

$R_1 - R_2 \to R_1$ $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$

$a = b = c = 0$

sys of gen $(0, 0, 0)$

Sys of gen of $U+V$:

$\{(-1,1,0),(0,0,1);(-3,1,1)\}$

3) $\{(-1, 1, 0), (0, 0, 1), (-3, 1, 1)\}$

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 \\ -3 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad R_2 + 3R_1 \to R_2 \quad \begin{bmatrix} 1 & -1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_2 / -2 \quad \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & \frac{1}{-2} \\ 0 & 0 & 1 \end{bmatrix}$$

$r = 3$ = nb of vectors ;

So the sys of gen of $U+V$ is

a sys of gen of $\mathbb{R}^3$

4) Let $V = (a, b, c) \in \mathbb{R}^3$

1) $U + V = R^3$ (because sys of gen of $U+V$ = sy of gen of $\mathbb{R}$

2) $U \cap W = (0, 0, 0) = 0$

So $\Rightarrow \mathbb{R}^3 = U \oplus V$

Ex 9 Let $E = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R}) \; ; \; a = c = 0 \right\}$

$F = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R}) , \; b = d = 0 \right\}$

1) Show that E and F are a subspace of $M_2(\mathbb{R})$

2) Show that $M_2(\mathbb{R}) = E \oplus F$.

solution 1) E subspace : We have $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in E$

hence $E \neq \phi$. Let $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in E$ and

$x' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in E$ and $\alpha \in \mathbb{R}$ then $a = c = 0$ and $a' = c' = 0$

We have $X + X' = \begin{pmatrix} a+a' & b+b' \\ c+c' & d+d' \end{pmatrix}$ and $\alpha X = \begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{pmatrix}$

then $X + X' \in E$ and $\alpha X \in E$, hence E is a subspace of $M_2(\mathbb{R})$ over $\mathbb{R}$.

F subspace : We have $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in F$ hence $F \neq \phi$

let $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in F$ and $Y = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in F$ and $\alpha \in \mathbb{R}$

then $b = d = 0$ and $b' = d' = 0$

We have $X + X' = \begin{pmatrix} a+a' & b+b' \\ c+c' & d+d' \end{pmatrix}$ and $\alpha X = \begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{pmatrix}$

$b + b' = d + d' = 0 + 0 = 0$ and $\alpha b = \alpha d = \alpha . 0 = 0$

then $X+X' \in F$ and $\alpha X \in F$ hence F is a subspace of $M_2(\mathbb{R})$ over $\mathbb{R}$

2) Let $X \in M_2(\mathbb{R})$ then there exist $a, b, c, d \in \mathbb{R}$ such that $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we have

$X = \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix} + \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix} \in E$ and $\begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} \in F$

hence $M_2(\mathbb{R}) = E + F$

Let $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in E \cap F$ then $a = c = 0$ and $b = d = 0$

hence $x = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and so $E \cap F = \{0\}$.

whence $M_2(\mathbb{R}) = E \oplus F$.